Guillemin, Moment maps and Combinatorial Inv. of Hamiltonian T^n-actions

$$\mathbb{CP}^1 = \{\text{lines in } \mathbb{C}^2\} = (\mathbb{C}^2 - 0)/\mathbb{C}^\times = S^3(\lambda)/S^1$$

Complex viewpoint :

$$1 \to \mathbb{C}^\times \xrightarrow{\Delta} (\mathbb{C}^\times)^2 \longrightarrow \mathbb{C}^\times \longrightarrow 1$$

$$z \mapsto (z,z)$$

$$(\mathbb{C}^\times)^2 \curvearrowright \mathbb{C}^2$$

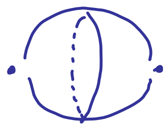
- $\Delta(\mathbb{C}^\times)$-action is free + proper on

$$\mathbb{C}^2_\Delta = \mathbb{C}^2 - 0 = (\mathbb{C}^\times \times 0) \cup (\mathbb{C}^\times \times \mathbb{C}^\times) \cup (0 \times \mathbb{C}^\times)$$

$\mathbb{C}^\times \times \mathbb{C}^\times$-orbits where $\Delta(\mathbb{C}^\times)$ acts freely.

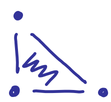
- $\mathbb{CP}^1 = \mathbb{C}^2_\Delta / \Delta(\mathbb{C}^\times)$

residue $\mathbb{C}^\times \curvearrowright \mathbb{CP}^1$



orbits : $(\mathbb{C}^\times \times 0)/\Delta\mathbb{C}^\times$, $(\mathbb{C}^\times \times \mathbb{C}^\times)/\Delta\mathbb{C}^\times$, $(0 \times \mathbb{C}^\times)/\Delta\mathbb{C}^\times$

- $\mathbb{C}^\times \overset{\Delta}{\hookrightarrow} (\mathbb{C}^\times)^{n+1} \curvearrowright \mathbb{C}^{n+1} \rightsquigarrow \mathbb{CP}^n$



$\mathbb{C}^\times \hookrightarrow (\mathbb{C}^\times)^{n+1}$ wt. proj. space

$z \mapsto (z^{d_1}, \dots, z^{d_{n+1}}) \rightsquigarrow \mathbb{WP}^n(d_1, d_2, \dots, d_{n+1})$

- $(\mathbb{C}^\times)^2 \hookrightarrow (\mathbb{C}^\times)^4 \curvearrowright \mathbb{C}^4$

$\Delta \times \Delta \rightsquigarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

others $\rightsquigarrow$ X^2 

- toric variety $(\mathbb{C}^\times)^n \curvearrowright X$, $(\mathbb{C}^\times)^n \curvearrowright (\mathbb{C}^\times)^n$ std., $(\mathbb{C}^\times)^n \subseteq X$ open dense

• Symplectic viewpoint. $\mathbb{CP}^1 = S^3/S^1$ $\quad(\mathbb{CP}^n)$

$\mathbb{C}^2$, $\omega_{\mathbb{C}^2} = i\,(dz_1\wedge d\bar{z}_1 + dz_2\wedge d\bar{z}_2)$ sympl.

$$= dr_1^2\wedge d\theta_1 + dr_2^2\wedge d\theta_2$$

On $r_1^2 + r_2^2 = \lambda$, i.e. $S^3(\sqrt{\lambda}) \subseteq \mathbb{C}^2$

$\Rightarrow dr_2^2 = -dr_1^2 \Rightarrow \omega = dr_1^2 \wedge d(\theta_1 - \theta_2)$

$\Rightarrow$ degenerate along $\theta_1 - \theta_2 \equiv$ const., i.e. $S^1 \overset{\Delta}{\hookrightarrow} S^1\times S^1 \hookrightarrow S^3$

ΔS^1-orbit

On $S^3(\sqrt{\lambda})/\Delta S^1$, ω descend to non-degen.

[Quotient by cpt gp, no non-Hausdorff issue]

• On $\mathbb{CP}^1 = S^3/S^1$.

coord. $z = z_2/z_1$ (i.e. $\theta = \theta_2 - \theta_1$, $r^2 = \frac{r_2^2}{r_1^2} = \frac{\lambda - r_1^2}{r_1^2}$)

$\Rightarrow \omega_{\mathbb{CP}^1_\lambda} = \lambda\,\partial\bar{\partial}\log(1+|z|^2) = \lambda\,\partial\bar{\partial}\log(|z_1|^2+|z_2|^2)$

• Why on $S^3 \subseteq \mathbb{C}^2$?

$S^1 \curvearrowright \mathbb{C}^2$ $\quad (e^{i\theta}z_1, e^{i\theta}z_2) \rightsquigarrow X = \frac{\partial}{\partial\theta_1} + \frac{\partial}{\partial\theta_2}$

$\Rightarrow \iota_X\omega = d(\underbrace{r_1 + r_2 - \lambda}_{\mu})$ $\quad \mu: \mathbb{C}^2 \to \mathbb{R}$ moment map

$\Rightarrow S^3 = \mu^{-1}(0)$

Recall $G \curvearrowright (M,\omega) \xrightarrow{\Phi} \mathfrak{g}^*$ $\quad$ G-equiv.

Hamiltonian action

$\phi^\xi : M \xrightarrow{\Phi} \mathfrak{g}^* \xrightarrow{\xi\lrcorner} \mathbb{R}$, $\quad \xi^\# \lrcorner\, \omega + d\phi^\xi = 0$

- G semi-simple $\Rightarrow$ Φ unique
- G Abelian $\Rightarrow$ $\Phi + c$ unique (up to $c \in \mathfrak{g}^*$)

• Eg. coadj. orbit $\mathcal{O}_\xi \subseteq \mathfrak{g}^*$ $\quad (S^2 \subseteq \mathbb{R}^3)$

($\equiv$ transitive Hamil. G^{cpt}-space, i.e. building blocks)

· Symplectic quotient / reduction

If $G \curvearrowright \Phi^{-1}(0)$ free (or $\Phi^{-1}(\mathcal{O}_\xi)$)

then $\omega|_{\Phi^{-1}(0)}$ decend to $\Phi^{-1}(0)/G = M/\!/G$

Abelian case : $T^s \curvearrowright (X^{2n}, \omega) \xrightarrow{\mu} \mathbb{R}^{s*}$

- (Atiyah, G.-S.) Convexity

  $\Delta := \mu(X)$ convex polytope, vertex $(\Delta) = \mu(X^T)$

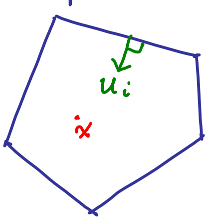
- effective T^s action $\Rightarrow$ $s \leq n$

- $s = n \xRightarrow{\text{Delzant}} \Delta$ determines everything!

$$1 \to N \to \underset{\mathbb{C}^d}{T^d} \to T^n \to 0$$

$\Rightarrow$ $X = \mathbb{C}^d /\!/ N$ w/ residue T^n-action

§ Appendix 1 (A 1. 1)



$\Delta \subseteq \mathbb{R}^n$ $\mathbb{Q}$ - convex polytope

$\langle x, u_i \rangle \geq \lambda_i$ $\quad i = 1, 2, \dots, d$

$u_i \in \mathbb{Z}^n_{prim}$, $\lambda_i \in \mathbb{Q}$, $d = \#$ facet

Assume : Every vertex is 'standard'

i.e. $\forall$ vertex p, $\exists$ n edges
s.t. up to $SL(n, \mathbb{Z})$, $p = 0$ & edges = coordi. axis.

Aim: Constr. $T^n \curvearrowright (X^{2n}_\Delta, \omega) \xrightarrow{\mu} \Delta \subseteq \mathbb{R}^n$

Indeed $X = \underbrace{\mathbb{C}^d / (\mathbb{C}^\times)^{d-n}}_{\text{Complex}} = \underbrace{\mathbb{C}^d /\!/ T^{d-n}}_{\text{Sympl.}}$

- Symplectic quotient / reduction

  If $G \curvearrowright \Phi^{-1}(0)$ free (or $\Phi^{-1}(\mathcal{O}_\xi)$)

  then $\omega|_{\Phi^{-1}(0)}$ decend to $\Phi^{-1}(0)/G = M/\!/G$

Abelian case : $T^s \curvearrowright (X^{2n}, \omega) \xrightarrow{\mu} \mathbb{R}^{s*}$

- (Atiyah, G.-S.) Convexity

  $\Delta := \mu(X)$ convex polytope, vertex $(\Delta) = \mu(X^T)$

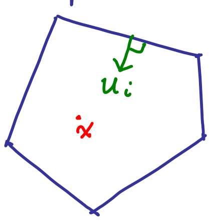
- effective T^s action $\Rightarrow$ $s \leq n$

- $s = n \xRightarrow{\text{Delzant}} \Delta$ determines everything!

$$1 \to N \to \underset{\mathbb{C}^d}{\overset{\curvearrowright}{T^d}} \to T^n \to 0$$

$\Rightarrow$ $X = \mathbb{C}^d /\!/ N$ w/ residue T^n-action

§ Appendix 1 (A1.1)



$\Delta \subseteq \mathbb{R}^n$ $\mathbb{Q}$ - convex polytope

$$\langle x, \underset{\mathbb{Z}^n_{prim}}{\underset{\Cap}{u_i}} \rangle \geq \underset{\mathbb{Q}}{\underset{\Cap}{\lambda_i}} \qquad i = 1, 2, \ldots, \underset{\#\text{facet}}{\underset{\shortparallel}{d}}$$

Assume : Every vertex is 'standard'

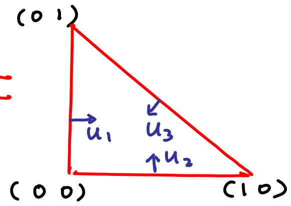
i.e. $\forall$ vertex p, $\exists$ n edges

s.t. up to $SL(n, \mathbb{Z})$, $p = 0$ & edges = coordi. axis.

Aim: Constr. $T^n \curvearrowright (X^{2n}_\Delta, \omega) \xrightarrow{\mu} \Delta \subseteq \mathbb{R}^n$

Indeed $X = \underset{\text{Complex}}{\mathbb{C}^d / (\mathbb{C}^\times)^{d-n}} = \underset{\text{Sympl.}}{\mathbb{C}^d /\!/ T^{d-n}}$

Eg. $\Delta =$



then $\mathbb{CP}^2 \overset{?}{=} \mathbb{C}^3/\mathbb{C}^\times \overset{?}{=} S^5/S^1$

$$\langle x, \begin{pmatrix}1\\0\end{pmatrix}\rangle \geq 0$$

$$\langle x, \begin{pmatrix}0\\1\end{pmatrix}\rangle \geq 0$$

$$\langle x, \begin{pmatrix}-1\\-1\end{pmatrix}\rangle \geq -1$$

$$0 \to \underset{\substack{\cup\\ e_1+e_2+e_3}}{\mathbb{R}} \longrightarrow \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \to 0$$

$$e_1 \longmapsto \begin{pmatrix}1\\0\end{pmatrix}$$

$$e_2 \longmapsto \begin{pmatrix}0\\1\end{pmatrix}$$

$$e_3 \longmapsto \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$\rightsquigarrow \quad \mathbb{C}^3/\mathbb{C}^\times = \mathbb{CP}^2$$

$$0 \to \underset{\substack{\parallel\\ \text{Kernel}}}{n} \longrightarrow \mathbb{R}^d \longrightarrow \mathbb{R}^n \longrightarrow 0$$

$\because \Delta$ bounded

$$e_i \longmapsto u_i$$

Can replace $\mathbb{R}$ by $\mathbb{Z}$, $\mathbb{C}$, $T_{\mathbb{C}} = \mathbb{C}/\mathbb{Z}$.

$$0 \to N_{\mathbb{C}} \longrightarrow \underset{\substack{\parallel\\ \mathbb{C}^d/2\pi i\mathbb{Z}^d}}{T_{\mathbb{C}}^d} \longrightarrow T_{\mathbb{C}}^n \longrightarrow 0$$

$$N_{\mathbb{C}} \leq T_{\mathbb{C}}^d \curvearrowright \mathbb{C}^d$$

Find where the action is free & proper.

Answer: $\mathbb{C}^d_\Delta = \bigcup_F \mathbb{C}^d_F \subseteq \mathbb{C}^d$

$\mathbb{C}^d_F = \{ z_i = 0 \Leftrightarrow i \in I \} \subseteq \mathbb{C}^d$ $\forall F$: codim r face in Δ
$F = \{ \langle x, u_i \rangle = \lambda_i \}_{i \in I}$
$|I| = r$

$\Rightarrow N_{\mathbb{C}} \leqslant T^d_{\mathbb{C}} \curvearrowright \underset{\text{free, proper}}{\mathbb{C}^d_\Delta} \subseteq \mathbb{C}^d$

- $X \triangleq \mathbb{C}^d_\Delta / N_{\mathbb{C}}$ Compact (?)
- $T^n_{\mathbb{C}} = T^d_{\mathbb{C}} / N_{\mathbb{C}} \curvearrowright X$ (residue action).
- $T^n_{\mathbb{C}} \curvearrowright X$ orbits $= \mathbb{C}^d_F / N_{\mathbb{C}}$
  $\longleftrightarrow$ faces of Δ

Sympl. viewpt. (Delzant)

$$0 \longrightarrow N \longrightarrow T^d \longrightarrow T^n \longrightarrow 0$$

$$T^d \curvearrowright \left(\mathbb{C}^d, \tfrac{i}{2} \sum^d dz_k \wedge d\bar{z}_k\right) \xrightarrow{h} \frac{1}{2} \sum^d |z_k|^2 \underbrace{e^*_k}_{\downarrow\, \alpha^*_k} - \lambda \in \mathbb{R}^{d*} \to n^*$$

$$\mathbb{C}^d /\!/ N = Z/N$$

$Z := h^{-1}(0)$ $\lambda = \sum_{k=1}^d \lambda_k \alpha^*_k$

$= \left\{ (z_1, \dots, z_d) \mid \frac{1}{2} \sum^d |z_k|^2 \alpha^*_k = -\lambda \right\}$

residue: $T \curvearrowright (Z/N, \omega) \xrightarrow{\phi} \mathfrak{t}^* = \mathbb{R}^n$

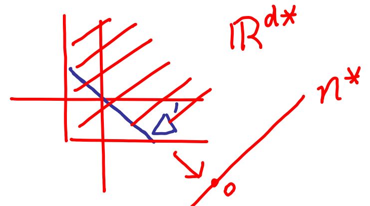
FACT: $\mathrm{Im}(\phi) = \Delta$

$$\Delta \subseteq \mathbb{R}^{n*} \rightsquigarrow X = Z/N \curvearrowleft T^n = T^d/N$$

$$\phi : X \longrightarrow \mathbb{R}^{n*} \rightsquigarrow \mathrm{Im}\,\phi \overset{?}{=} \Delta$$

- $J : \mathbb{C}^d \longrightarrow (\mathbb{R}^d)^*$, $(z_1, \dots) \mapsto \frac{1}{2}(|z_1|^2 + \lambda_1, \dots)$ $\Rightarrow \mathrm{Im} J = \left\{ \begin{array}{l} p \in \mathbb{R}^{d*} : i = 1, \dots, d \\ \langle p, e_i \rangle \geq \lambda_i \end{array} \right\}$

- $$\begin{array}{ccccccccc} 0 & \to & \mathbb{R}^{n*} & \to & \mathbb{R}^{d*} & \to & n^* & \to & 0 \\ & & \cup\!\shortmid & & \cup\!\shortmid & & & & \\ & x & \Delta & \xhookrightarrow{\sim} & \Delta' & p & & & \end{array}$$



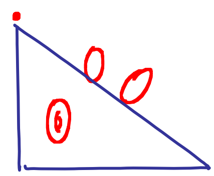
$$x \in \Delta \iff \underbrace{\langle x, u_i \rangle_{\mathbb{R}^n}}_{\langle p, e_i \rangle_{\mathbb{R}^d}} \geq \lambda_i \qquad i = 1, \dots, d$$

$$\begin{tikzcd} T^d \arrow[r, bend left] & \mathbb{C}^d \arrow[r, "J"] & \mathbb{R}^{d*} \arrow[d, "\iota^*"] \\ N \arrow[u, hook, "\iota"] & & n^* \end{tikzcd} \qquad \therefore \quad X_\Delta = \frac{(\iota^* \circ J)^{-1}(0)}{N} = \frac{J^{-1}(\Delta')}{N}$$

$$\begin{tikzcd} & & \mathbb{R}^{n*} \supseteq \Delta \quad x \arrow[d, hook] \\ T^d \arrow[r, bend left] & \mathbb{C}^d_z \arrow[r, "J"] & \mathbb{R}^{d*} \supseteq \Delta' \quad p \arrow[d, "\iota^*"] \\ N \arrow[u, hook, "\iota"] & & n^* \end{tikzcd} \qquad \Bigg| \quad X_\Delta = \frac{(\iota^* \circ J)^{-1}(0)}{N} = \frac{J^{-1}(\Delta')}{N}$$

$$x \in \Delta \iff \underbrace{\underbrace{\langle x, u_i \rangle_{\mathbb{R}^n}}_{\langle p, e_i \rangle_{\mathbb{R}^d}}}_{\frac{|z_i|^2}{2} + \lambda_i} \geq \lambda_i \quad \forall i \qquad \text{w/} \quad J(z) = p$$

$\Rightarrow$
- $\mathrm{Im}(\phi) = \Delta$ $\quad (\because J((\iota^* \circ J)^{-1}(0)) = \Delta')$
- $x \in \Delta^\circ$ $(\gneqq \lambda_i)$ $\Rightarrow$ $z_i \neq 0$ $\forall i$ $\Rightarrow$ $T^d \curvearrowright J^{-1}(p)$ simply transitive.



- $T^n \curvearrowright X_\Delta^{2n} \longrightarrow \Delta \subseteq \mathbb{R}^{n*}$ $\qquad X^T \longleftrightarrow \mathrm{Vertex}(\Delta)$

$$X_\Delta \triangleq \mathbb{C}^d_\Delta / N_\mathbb{C} \overset{?}{\cong} \underbrace{\mu^{-1}(0)/N}_{Z} \qquad \text{cpx/sympl.} \ (G_\mathbb{C}/B = G/T)$$

i.e. $\forall\, (N_\mathbb{C}\text{-orbit} \subseteq \mathbb{C}^d_\Delta) \cap Z \overset{?}{\leadsto} N\text{-orbit}$ $(\exists!)$

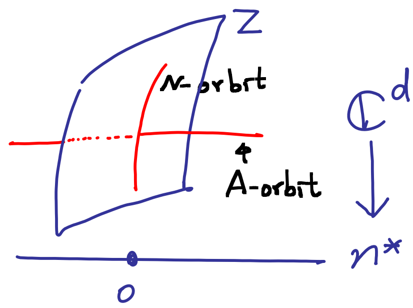
$$N_\mathbb{C} = N e^A \quad \text{w/} \quad A = N_\mathbb{C} \cap \mathbb{R}^d_+ \qquad (\underbrace{\mathbb{C}^\times = S^1 \cdot \mathbb{R}_+}_{\text{Iwasawa dec.}})$$

i.e. $\forall\, (A\text{-orbit } Y \text{ in } \mathbb{C}^d_\Delta) \cap Z \overset{?}{\leadsto} 1 \text{ pt.}$

i.e. by considering $Z_\lambda = \mu^{-1}(\lambda)$ for all $\lambda \in n^*$

$f|_Y : Y \longrightarrow n^*$ diffeo. onto (open convex) image

$$\begin{tikzcd} N \arrow[r, phantom, "\curvearrowright"] & \mathbb{C}^d \arrow[r, "\mu"] & n^* \\ & Y \arrow[u, phantom, "\cup|"] & \mu(Y) \arrow[u, phantom, "\cup|"] \end{tikzcd} \qquad Z = \mu^{-1}(0)$$



i.e. (enough) $\mu|_Y$ diffeo.? $\quad \mu(Y) \overset{\text{open convex?}}{\subseteq} n^*$

$$Y_1, Y_2 \subseteq \mathbb{C}^d_F \overset{?}{\Rightarrow} \mu(Y_1) = \mu(Y_2)$$

$$\begin{tikzcd} \mu : \mathbb{C}^d \arrow[rr] \arrow[dr] & & n^* \\ & \mathbb{R}^{d*} \arrow[ur, "\iota^*"] & \end{tikzcd} \qquad e_i \mapsto \alpha_i$$

$$(z_1, \dots, z_d) \mapsto \underbrace{\tfrac{1}{2}(|z_1|^2, \dots, |z_d|^2) + \lambda_0}_{\tfrac{1}{2}\sum |z_k|^2 e_k + \lambda_0} \mapsto \frac{1}{2}\sum_k |z_k|^2 \alpha_k + \lambda$$

$$\underset{\substack{N_\mathbb{C} \cap \mathbb{R}^d \\ y}}{A} \le N_\mathbb{C} \le (\mathbb{C}^\times)^d \curvearrowright \mathbb{C}^d$$

$$y \cdot (z_1, \dots, z_d) = (e^{\alpha_1(y)} z_1, \dots, e^{\alpha_d(y)} z_d)$$

$$\Rightarrow \mu(y \cdot z) = \frac{1}{2} \sum |z_i|^2 e^{2\alpha_i(y)} \alpha_i + \lambda \in n^*$$

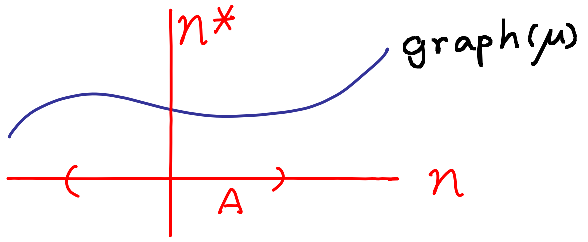
($f|_{A\text{-orbit}}$ diffeo. onto open convex $\subseteq n^*$?)

Fix $(z_1, \dots, z_d) \in \mathbb{C}^d_\Delta$ write $a_i = |z_i|^2 \geq 0$

$$\mu : A \longrightarrow \mathfrak{n}^*$$

$$\mu(y) = \frac{1}{2} \sum a_i e^{2\alpha_i(y)} \alpha_i + \lambda$$

like: $\mathbb{C}^\times = S^1 \cdot \mathbb{R}_{>0} = Ne^A$, $\mu : A \subseteq \mathfrak{n} \to \mathfrak{n}^*$



T^*A

$$\mu = dF \quad \text{where} \quad F : A \to \mathbb{R}$$

$$F(y) \stackrel{\Delta}{=} \frac{1}{4} \sum a_i e^{2\alpha_i(y)} + \lambda(y)$$

$$\nabla^2 F = 2 \sum a_i e^{2\alpha_i(y)} \alpha_i \otimes \alpha_i$$

$$z \in \mathbb{C}^d_F \subseteq \mathbb{C}^d_\Delta$$

$\Rightarrow$ $A \leqslant N_{\mathbb{C}} \curvearrowright \mathbb{C}^d$, freely at z. wt: $\alpha_1, \dots, \alpha_d$

$\Rightarrow$ $\{ \alpha_i \mid a_i = |z_i|^2 \gneq 0 \}$ span $\mathfrak{n}^*$

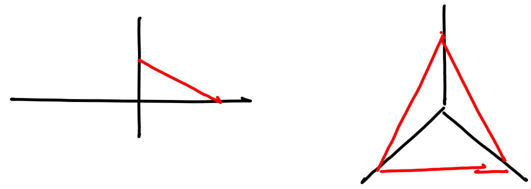
$(\because \ y \cdot (z_1, \dots, z_d) = (e^{\alpha_1(y)} z_1, \dots, e^{\alpha_d(y)} z_d)$

$\Rightarrow$ $\nabla^2 F > 0$, i.e. F is strictly <u>convex</u>.

$\xRightarrow[\text{transf.}]{\text{Legendre}}$ $\mu = dF$ diffeo. onto open convex set. QED.

Review: §1

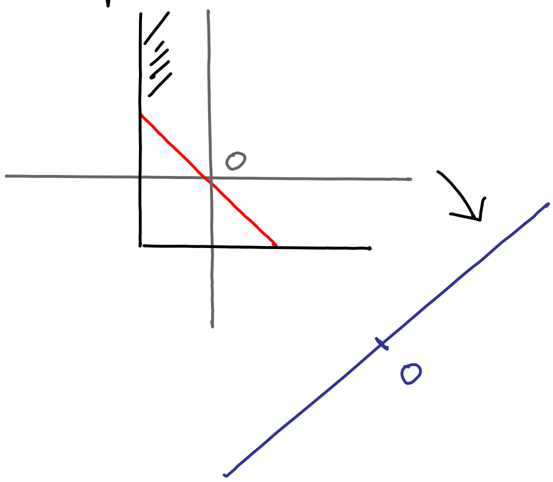
- $\Delta \subset \mathbb{R}^{n*}$



$\Rightarrow \quad \Delta \subseteq \mathbb{R}^{d*}_{\geq 0}$ $\qquad d = \#$ codim 1 faces of Δ

w/ $\partial\Delta = \Delta \cap \{\text{coordi hyperplanes}\}$

Up to translation, make $\Delta \ni 0 \in \mathbb{R}^{d*}$



$$0 \to \mathbb{R}^{n*} \xrightarrow{\iota^*} \mathbb{R}^{d*} \longrightarrow \mathfrak{n}^* \to 0$$

$\Delta \subseteq \mathbb{R}^{n*}$ described by $\langle \underset{\mathbb{R}^{n*}}{x}, \underset{\mathbb{R}^{n}}{u_i} \rangle \geq \lambda_i \qquad i = 1, \dots, d.$

$\rightsquigarrow$

$$\mathbb{Z}^d \xrightarrow{\begin{pmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{pmatrix}} \mathbb{Z}^n \qquad \mathfrak{n}_{\mathbb{Z}} = \ker$$

to quotient away!

$$0 \leftarrow \underset{u_i}{\mathbb{R}^n} \longleftarrow \underset{e_i}{\mathbb{R}^d} \longleftarrow \mathfrak{n} \longleftarrow 0$$

$\langle x, u_i \rangle \geq \lambda_i \qquad \langle p, e_i \rangle \geq \lambda_i$

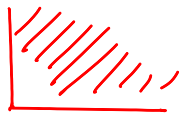
$$0 \longrightarrow \underset{x}{\mathbb{R}^{n*}} \longrightarrow \underset{p.}{\mathbb{R}^{d*}} \longrightarrow \mathfrak{n}^* \longrightarrow 0$$

$$\omega = \frac{i}{2}\, dz \wedge d\bar{z} = r\, dr \wedge d\theta = d\left(\frac{|z|^2}{2}\right) \wedge d\theta$$

Standard case:

$$\begin{tikzcd} T^d \arrow[r, phantom, "\curvearrowright"] & \mathbb{C}^d \arrow[r, "\mu"] & \mathbb{R}^{d*} \end{tikzcd}$$

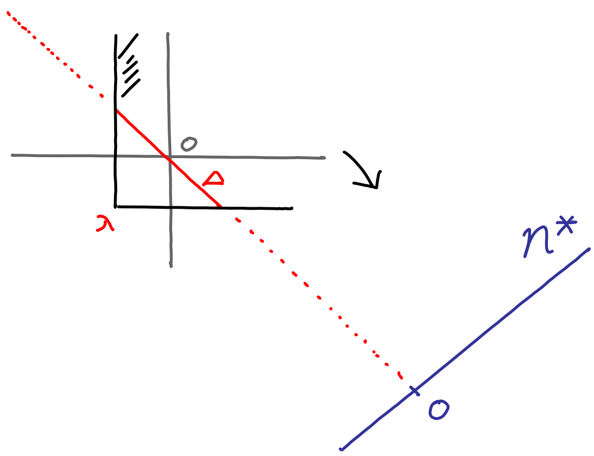
$$\begin{array}{ccc} (z_1, \dots) & \longmapsto & \left(\frac{|z_1|^2}{2} + \lambda_1, \dots\right) \\ \downarrow & & \\ (e^{i\theta_1} z_1, \dots) & & \end{array}$$

$$\mathrm{Im}(\mu) =$$ 

$$\langle p, e_i \rangle \geq \lambda_i$$

$$\begin{tikzcd} 0 & T^n \arrow[l] & T^d \arrow[l] \arrow[d, phantom, "\curvearrowright"] & N \arrow[l] & 0 \arrow[l] \\ & & \mathbb{C}^d \arrow[d] & & \\ 0 \arrow[r] & \mathbb{R}^{n*} \arrow[r] & \mathbb{R}^{d*} \arrow[r] & \mathfrak{n}^* \arrow[r] & 0 \end{tikzcd}$$

$$\mathbb{R}^{n*} \supseteq \Delta$$



$$\Longrightarrow \quad T^n \curvearrowright \underbrace{\mathbb{C}^d /\!/ N}_{X_\Delta} \xrightarrow{\ \mu\ } \mathbb{R}^{*} \supseteq \Delta$$

§ App. $N \curvearrowright \mathbb{C}^d \xrightarrow{\mu_N} \mathfrak{n}^*$

$$X_\Delta = \mathbb{C}^d /\!/ N = \{\mu_N = 0\} / N$$

Cpx viewpt. $N_{\mathbb{C}} = N e^{\mathfrak{n}} \curvearrowright \mathbb{C}^d$

$\mathbb{C}^\times = S^1 \, \mathbb{R}_+$

Want $X_\Delta =$ "$\mathbb{C}^d / N_{\mathbb{C}}$" $= \mathbb{C}^d_\Delta / N e^{\mathfrak{n}}$

- $T^d_{\mathbb{C}} = (\mathbb{C}^\times)^d \curvearrowright \mathbb{C}^d$ orbits : $\mathbb{C}^\times \times 0 \times \mathbb{C}^\times \times \mathbb{C}^\times$'s

  free : only 1 : $(\mathbb{C}^\times)^d \subseteq \mathbb{C}^d$

$N_{\mathbb{C}} \leq T^d_{\mathbb{C}} \curvearrowright \mathbb{C}^d$ free on $\mathbb{C}^d_F$'s $F \subseteq \Delta$ face

$$\mathbb{C}^d \supseteq \mathbb{C}^d_\Delta = \coprod_F \mathbb{C}^d_F$$

$$\mathbb{C}^d_\Delta / N e^{\mathfrak{n}} \overset{?}{=} \{\mu = 0 \in \mathfrak{n}^*\}/N$$

$\forall z \in \mathbb{C}^d_\Delta$, $(e^{\mathfrak{n}} \cdot z) \cap \{\mu = 0\}$ 1 pt ?

$$\varphi : \mathfrak{n} \xrightarrow{\exp} e^{\mathfrak{n}} \xrightarrow{\cdot z} \mathbb{C}^d_\Delta \xrightarrow{\mu} \mathfrak{n}^*$$

Claim: φ diffeo onto $\mathrm{Im}\,\varphi \underset{\text{open convex}}{\subseteq} \mathfrak{n}^*$

$$y \mapsto (\alpha_1(y), \dots) \qquad \alpha_1 \mapsfrom e_1$$

$$\begin{tikzcd}
\mathfrak{n} \arrow[d, hook] \arrow[rrr, bend left, "\varphi"] & & & \mathfrak{n}^* \\
\mathbb{R}^d \arrow[r, "\exp"] & \mathbb{R}^d_+ \arrow[r, "\cdot z"] & \mathbb{C}^d_\Delta \arrow[r, "\mu"] & \mathbb{R}^{d*} \arrow[u, two heads]
\end{tikzcd}$$

$$(x_1, \dots) \qquad (e^{x_1}, \dots) \qquad (e^{x_1} z_1, \dots) \qquad \left(\tfrac{1}{2}\underbrace{|z_1|^2}_{a_1} e^{2x_1} + \lambda_1, \dots\right)$$

$$\Longrightarrow \quad \varphi(y) = \frac{1}{2}\sum_{i=1}^{d} a_i\, e^{2\alpha_i(y)} \alpha_i + \bar{\lambda} \qquad a_i = |z_i|^2 \geq 0$$

Treat $\varphi \in \Omega^1(\mathfrak{n})$, $\varphi = dF$

$F : \mathfrak{n} \longrightarrow \mathbb{R}$ $\qquad F(y) = \frac{1}{4}\sum_{i=1}^{d} a_i\, e^{\alpha_i(y)} + \bar{\lambda}(y)$

- $z \in \mathbb{C}^d_\Delta \Rightarrow$ enough $|z_i|^2 > 0 \Rightarrow \nabla^2 F > 0 \Rightarrow$ ✓

§2 Duistermaat-Heckman Thm.

$$S^1 \circlearrowright (M^{2n}, \omega) \xrightarrow{\mu} \mathbb{R} \ni a \qquad (\text{Same for } \underset{\sim}{T} \circlearrowright (M,\omega))$$

Assume: a regular value of J.

$(X_a := \underbrace{\mu^{-1}(a)}_{Z}/S^1, \nu_a)$ sympl. quotient

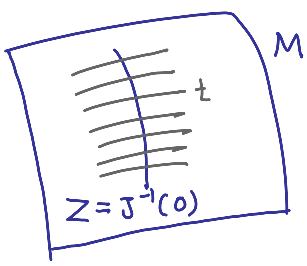
$$S^1 \longrightarrow Z \xrightarrow{\pi} X_a \qquad S^1\text{-bdl.}$$

$$\omega|_Z = \pi^* \nu_a$$

Recall $\quad \underset{\theta\ ,\ t}{S^1 \times \mathbb{R}} \qquad \omega = dt \wedge d\theta = d(t \underset{d\theta}{\underbrace{\alpha}})$

α ANY conn. 1-form on Z.

On $Z \times \mathbb{R}$, $\pi^* \nu_a + d(t\alpha)$ sympl. (for small t)



Assume: a regular value of μ

Equivar. Darboux : near $\mu^{-1}(0)$

$$(M, \omega) \overset{S^1\text{-eq.}}{=\!=} (Z \times \mathbb{R}, \pi^* \nu_0 + d(t\alpha))$$

$\exists$ connection $\alpha \in \Omega^1(Z)^{S^1}$

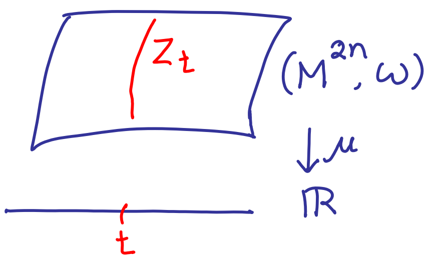
on $S^1 \to Z \xrightarrow{\pi} X_0$

$\Rightarrow (X_a, \nu_a) \cong (X_0, \nu_0 + aF)$ w/ $F = d\alpha$ curv.

$$\Rightarrow \quad Vol(X_a, \nu_a) = \int_{X_a} e^{\nu_a} = \int_{X_0} e^{\nu_0 + aF} = \int_{X_0} \frac{(\nu_0 + aF)^{n-1}}{(n-1)!}$$

poly. in a of $\deg = n-1$.

check



Around reg. values

$$\omega^n = (\pi^* \nu_0 + d(t\alpha))^n$$

$$= \pi^* (\nu_0 + tF)^{n-1} \alpha \wedge dt$$

Push-forward measure : $m_{DH} = \mu_*(\omega^n)$

$$m_{DH} = f(t)\,dt \quad \text{on} \quad \mathbb{R} \qquad (n! = 1)$$

$$f(t) = \int_{Z_t} \frac{\omega^n}{n!} \Big/ dt \qquad \int_{Z_t} \frac{\pi^*(\nu_0 + tF)^{n-1}\,\alpha}{n!}$$

$$= \frac{1}{n}\int_{X_t} \frac{(\nu_0 + tF)^{n-1}}{(n-1)!} = \frac{1}{n}\,\mathrm{Vol}(X_t). \qquad \text{poly. in } t$$

Around critical values ?

~ Morse theory, symplectic surgery (blowup)

Topo: change

$S^1 \curvearrowright (M,\omega) \xrightarrow{\mu} \mathbb{R} \ni 0$ critical value

$S^1 \curvearrowright Z = \mu^{-1}(0)$

'assume'

isotropy group: $\{1\}$, $\cancel{\mathbb{Z}/m}$, S^1

(free) (fix)

$$Z/S^1 = Z_{free}/S^1 \sqcup Z_{fix}.$$

Say Z_{fix} 1 point.

Darboux $\Rightarrow$ std. model $S^1 \curvearrowright \mathbb{C}^{p,q} \xrightarrow{\mu} \mathbb{R}$

$(e^{i\theta}z_1, e^{-i\theta}z_2)$ $\quad \dfrac{|z_2|^2 - |z_1|^2}{2}$

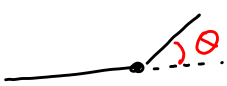
$$Z_0 = \{|z_1|^2 = |z_2|^2\} = Z_{free} \sqcup \underbrace{Z_{fix}}_{\{0\}}$$

$$X_0 = Z_0/S^1 = C\Big(\underbrace{\frac{S^{2p-1} \times S^{2q-1}}{S^1}}\Big)$$

$$\mathbb{C}P^{p-1} \xleftarrow{S^{2q-1}} \qquad \xrightarrow{S^{2p-1}} \mathbb{C}P^{q-1}$$

$\rightsquigarrow$ 2 ways to resolve cone sing.: replace 0 by $\mathbb{C}P^{p-1}$ or $\mathbb{C}P^{q-1}$

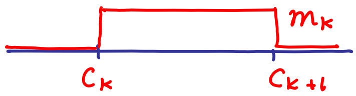
$\rightsquigarrow$ $X_{-\varepsilon}$ and $X_{+\varepsilon}$ (flop)

$\mathrm{Vol}(X_\varepsilon)$:  $\tan(\theta) \sim [\text{exc. locus}]$.

Exact stationary phase : $\phi : M \longrightarrow \mathbb{R}$

$$\underbrace{(2\pi i)^{-n}\int_M e^{i\lambda\phi}\,\frac{\omega^n}{n!}}_{\mathcal{F}\left(\underbrace{m_{DH}}_{\sum p_k(t)\, m_k}\right)\cdot(\text{const}).} \;\underset{\lambda \gg 0}{=}\; \frac{1}{\lambda^n}\sum_{\substack{c\\ \text{cr.pt.}}} e^{i\lambda\phi(c)}\,\frac{1}{\prod k_i(c)} + O(\lambda^{-\infty})$$

Fourier transf.



p_k poly. deg $(n-1)$

$$\sum p_k\left(\frac{1}{i}\frac{\partial}{\partial\lambda}\right)\underbrace{\mathcal{F}(m_k)}_{\frac{1}{i\lambda}\left(e^{i\phi(c_{k+1})\lambda} - e^{i\phi(c_k)\lambda}\right)}$$

$\Rightarrow$ no nontrivial term of order $O(\lambda^{-\infty})$

$\Rightarrow$ exact formula !!

- $\phi(c_k)$'s , $\prod k_r(c_k)$'s $\xrightarrow[\text{formula}]{\text{above}}$ p_k $\Longrightarrow$ m_{DH} completely !

§3 $(\underbrace{T^*\mathbb{R}^n}_{M}, \omega_{can}) \rightsquigarrow$ Hilbert space $Z_M = L^2(\mathbb{R}^n)$

Quantization

(M, ω) $\overset{?}{\rightsquigarrow}$ Z_M (unique projectively)

$\omega / \mathbb{Z}$

$\mathrm{Vol}(M) = \int_M e^\omega$ $\qquad$ $\dim Z_M = \int_M e^\omega \, Td_M$

$\overset{\text{R.R.}}{=}$ Index $\bar\partial$

Recall : $Td_M = \prod_{i=1}^{n} \frac{x_i}{1-e^{-x_i}}$

x_i : Chern root of T_M

When $M = X_\Delta$ Toric

$$\left.\begin{array}{l} N \hookrightarrow T^d \curvearrowright \mathbb{C}^d \\ \downarrow \\ Z = \mu^{-1}(0) \\ \downarrow \\ X_\Delta = Z/N \end{array}\right\} \begin{array}{l}\text{line bdl} \\ \bigoplus_{i=1}^d L_i = T_{X_\Delta} \\ \text{(stably)}\end{array} \Bigg\} \Rightarrow Td_{X_\Delta} = \prod_{i=1}^{d} \frac{c_1(L_i)}{1-e^{-c_1(L_i)}}$$

$$\Rightarrow \dim Z_{X_\Delta} = \int_{X_\Delta} e^{\omega_\Delta} \cdot Td_{X_\Delta} = \int_{X_\Delta} e^{\omega_\Delta} \cdot \prod_{i=1}^{d} \frac{c_1(L_i)}{1-e^{-c_1(L_i)}}$$

$$= Td\left(\tfrac{\partial}{\partial h_1}, \dots, \tfrac{\partial}{\partial h_d}\right) \underbrace{\left(\int_{X_\Delta} e^{\omega_\Delta + \sum_{i=1}^{d} h_i c_1(L_i)} \right)}\Bigg|_{h=0}$$

$$= Td\left(\tfrac{\partial}{\partial h}\right) \cdot Vol(\Delta_h) \Big|_{h=0}$$

$$(M, \omega/\mathbb{Z}) \stackrel{?}{\rightsquigarrow} Z_M$$

(axioms)

- $Z_{M_1 \times M_2} = Z_{M_1} \otimes Z_{M_2}$, $Z_{\overline{M}} = Z_M^*$
- $G \curvearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^* \rightsquigarrow G \curvearrowright Z_M$ "Quantizat^{n} commutes w/ reductions"

$$\begin{array}{ccc} G \curvearrowright (M,\omega) \xrightarrow{\mu} \mathfrak{g}^* & \rightsquigarrow & G \curvearrowright Z_M \\ \wr\downarrow & & \wr\downarrow \\ (M/\!/G, \omega_{red}) & \rightsquigarrow & Z_{M/\!/G} = (Z_M)^G \end{array}$$

Note : $M/\!/G = \mu^{-1}(0)/G$

$$M/\!/_{\xi} G = \mu^{-1}(\mathcal{O}_\xi)/G = \mu^{-1}_{M \times \overline{\mathcal{O}_\xi}}(0)/G$$

$$Z_{M/\!/_\xi G} = (Z_{M \times \overline{\mathcal{O}_\xi}})^G = Hom(Z_{\mathcal{O}_\xi}, Z_M)^G$$

Borel-Weil-Bott (G: cpt). $\mathcal{O}_\xi \longleftrightarrow Z_{\mathcal{O}_\xi}$

Coadj orbit/$\mathbb{Z}$ — irred. rep. of G — multi. of corresp. irred. rep.

Strategy : $\underset{\text{(explicit)}}{Z_{\mathbb{C}^d}}$ + $X_\Delta = \mathbb{C}^d /\!/_\lambda N \Longrightarrow Z_{X_\Delta}$

Config. Sp: $(M, \omega/\mathbb{Z})$ mfd. $\rightsquigarrow$ Z_M Hilbert Space

Observable: $f \in C^\infty(M)$ $\rightsquigarrow$ $S_f : Z_M \to Z_M$ self-adj op.

$$\mathbb{C} \to L \to M \qquad \operatorname{curv}(\nabla) = \omega$$

$Z_M \subseteq \Gamma_{L^2}(M, L)$ polarized

$\xi \in V \leq T_M \otimes \mathbb{C}$ Lagr., integrable

$\nabla_\xi s = 0$

(eg. $G \curvearrowright M \to \mathfrak{g}^*$)

$f \in C^\infty(M)$ $\qquad \nabla_{X_f} + 2\pi i f : \Gamma(M, L) \circlearrowleft$

Example: $T^d \curvearrowright T^*\mathbb{R}^d = \mathbb{C}^d$, $\quad \omega = \frac{i}{2} \sum dz_k \wedge d\bar{z}_k$

$L = \mathbb{C}^d \times \mathbb{C}$, $\quad \nabla = d + \underbrace{\alpha}_{\frac{i}{2} \sum z_k d\bar{z}_k}$

$\Gamma(M, L) \ni s \longleftrightarrow 1 \qquad \nabla s / s = 2\pi i \alpha$

$(L, \nabla, \langle\ \rangle) \quad \nabla(\langle\ \rangle) = 0$

- At $z \in M = \mathbb{C}^d$, $\langle f_1 s, f_2 s \rangle = e^{-2\pi |z|^2} f_1 \bar{f_2}$

$\Rightarrow \Gamma_{L^2}(M, L) \ni f s$ s.t. $\left(\frac{i}{2}\right)^n \int |f|^2 e^{-2\pi |z|^2} dz\, d\bar{z} < \infty$

- Polarization: $\langle \frac{\partial}{\partial z_k} \text{'s} \rangle \leq T_M \otimes \mathbb{C}$

$\nabla_{\frac{\partial}{\partial z_k}} = \frac{\partial}{\partial z_k} + \overbrace{\underbrace{\alpha}_{z\, d\bar{z}}\left(\frac{\partial}{\partial z_k}\right)}^{0} \Rightarrow f s \in \Gamma_{pol.}(M, L)$ iff $\partial f = 0$

- $T^d \curvearrowright \underbrace{\Gamma_{L^2 + pol.}(M, L)}_{Z_{\mathbb{C}^d}}$ $\quad$ wt vector: $\bar{z}_1^{m_1} \cdots \bar{z}_d^{m_d} s$ multi = 1 w/ $m_i \geq 0 \ \forall i$

• $0 \to N \to T^d \xrightarrow{\pi} T^n \to 0$ (← via $u_1, \dots, u_d \in \mathbb{Z}^n$) $\qquad 0 \to \mathbb{Z}^{n*} \xrightarrow{\pi^*} \mathbb{Z}^{d*} \longrightarrow n_{\mathbb{Z}}^* \to 0$

$$\lambda = (\lambda_1, \dots, \lambda_d) \mapsto \lambda_0$$

$$\underbrace{Z\,\mathbb{C}^d /\!/_{\lambda_0} N}_{X_\Delta} \overset{\text{Fact}}{=\!=} (Z\,\mathbb{C}^d)_{-\lambda_0} \quad \text{(N-wt.)}$$

$-\lambda_0$: N-wt in $Z\mathbb{C}^d$

$\longleftrightarrow -\lambda + \pi^* y : T^d$-wt in $Z\mathbb{C}^d$, $y \in \mathbb{Z}^{n*}$

i.e. $\in \mathbb{Z}^{d*}_{\geq 0}$

$$\underbrace{\langle -\lambda + \pi^* y, e_i \rangle}_{-\lambda_i + \langle y, \underbrace{\pi(e_i)}_{u_i} \rangle} \geq 0 \qquad i = 1, \dots, d$$

i.e. $\langle y, u_i \rangle \geq \lambda_i$, $\forall i$

i.e. $y \in \Delta \cap \mathbb{Z}^{n*} =: \Delta_{\mathbb{Z}}$

$\Rightarrow \dim Z_{X_\Delta} = \# \Delta_{\mathbb{Z}}$.

§4. $\# \Delta_{\mathbb{Z}} \overset{?}{=\!=} \mathrm{Td}\left(\frac{\partial}{\partial h}\right) \mathrm{Vol}(\Delta_h) \Big|_{h=0}$

$\Delta_{\mathbb{Z}} \subseteq \mathbb{Z}^{n*} \subseteq \mathbb{R}^{n*}$

In fact, (Khovanskii) $\quad m_\Delta^{\mathbb{Z}}$ "=" $\mathrm{Td}\left(\frac{\partial}{\partial h}\right) m_{\Delta_h} \Big|_{h=0}$

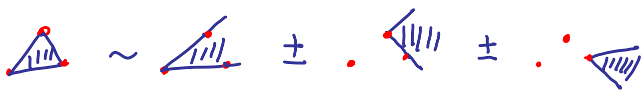
namely, $\forall$ polynomial f,

$$\sum_{x \in \Delta_{\mathbb{Z}}} f(x) = \mathrm{Td}\left(\frac{\partial}{\partial h}\right) \int_{\Delta_h} f(x)\, dx \Bigg|_{h=0}$$

Pf: 1° $\Delta = (-\infty, 0]$ (later)

2° $\Delta = [2, 5] = (-\infty, 5] \setminus (-\infty, 2]$, $\quad \mathrm{Td}(-x) = e^{-x}\mathrm{Td}(x) \Rightarrow \checkmark$ (fcl. eqt)

3° General case Δ $\quad$ (need $m^S_\Delta = \sum (-1)^q m^S_{\Delta_q^\#}$ later)



Pf of 1° : (Euler - MacLaurin formula)

$$\sum_{n\in\mathbb{Z}_{\le 0}} f(n) = Td\left(\frac{\partial}{\partial h}\right)\int_{-\infty}^{h\,?} f(x)\,dx\,\Big|_{h=0} \qquad f: \text{poly.}$$

(i) $f(x) = e^{ax}$, $a>0$ (NOT poly. !)

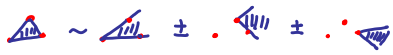
$$\text{R.H.S.} = \underbrace{Td\left(\frac{\partial}{\partial h}\right)}_{Td=\frac{y}{1-e^{-y}}} \underbrace{\int_{-\infty}^{h} e^{ax}\,dx}_{e^{ah}/a}\,\Big|_{h=0} = \frac{1}{a}\,Td(a)\,e^{ah}\Big|_{h=0}$$

$$= \frac{1}{a}\cdot\frac{a}{1-e^{-a}} = \sum_{n=0}^{\infty} e^{-an} = \text{L.H.S.}$$

(ii) $f = \sum_{i=1}^{k} p_i(x)\,e^{a_i x}$, p_i poly. $a_i > 0$ $\left(\because\ \frac{d}{da}(i) \Rightarrow \checkmark\right)$

(iii) okay for $a_i > 0 \Rightarrow$ okay $a_i \in \mathbb{R}$

i.e. $\#\Delta_{\mathbb{Z}} = Td\left(\frac{\partial}{\partial h}\right) Vol(\Delta_h)\Big|_{h=0}$ & $m_\Delta^{\mathbb{Z}} = Td\left(\frac{\partial}{\partial h}\right) m_{\Delta_h}\Big|_{h=0}$ #

Still need $m_\Delta^S \overset{?}{=} \sum (-1)^q\, m^S_{\Delta_q^\#}$ 

By refining $\underset{\downarrow 0}{\varepsilon}\mathbb{Z}^n \subseteq \mathbb{R}^n$ $\Downarrow ?$ $m_\Delta \overset{?}{=} \sum(-1)^q\, m_{\Delta_q}$

Warmup:

$$S^1 \curvearrowright \Big(\mathbb{C}, \underbrace{\underbrace{\omega}_{\frac{i}{2}dz\,d\bar z}}_{ds\wedge d\theta}\Big) \xrightarrow{\mu} \mathbb{R}$$

$$z \longmapsto s = \frac{|z|^2}{2}$$

$$\Rightarrow m_{DH} = \mu_*(\omega) = \Big(\underbrace{\quad}_{\chi_{[0,\infty)}}\Big)\cdot ds$$

Similar, $T^d \curvearrowright \mathbb{C}^d \xrightarrow{\mu} \mathbb{R}^d$

$$m_{DH} = \chi_{\mathbb{R}^d_{\ge 0}} \cdot m_{Leb.}$$

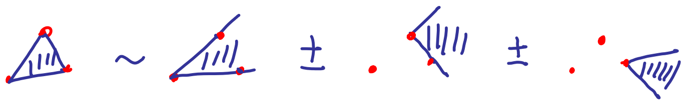
Similar, $N \leq T^d \curvearrowright \overset{\alpha_1,\dots,\alpha_d \,:\, N\text{-wt.}}{\mathbb{C}^d} \longrightarrow \mathbb{R}^{d*} \longrightarrow \mathfrak{n}^*$

on $\mathfrak{n}^*$, $\underbrace{m_{DH} = \mathrm{Vol}\left(\left\{(s_1,\dots,s_d)\in\mathbb{R}^d_+ \;\middle|\; \Sigma\, s_i\alpha_i = \overset{\mathfrak{n}^*}{\overset{\Psi}{\lambda}}\right\}\right)}_{\text{discrete version: Kostant partition fn.}} dm_{\text{Leb.}}$

$\lambda \in \mathfrak{n}^*_{\mathbb{Z}}$, $N(\lambda) = \#\left\{(n_1,\dots,n_d)\in\mathbb{Z}^d_{\geq 0} \;\middle|\; \Sigma\, n_i\alpha_i = \lambda\right\}$

Toric: $T^n \curvearrowright X^{2n}_\Delta \xrightarrow{\ \phi\ } \Delta \subseteq \mathbb{R}^n$

Recall: on $\mathbb{R}^n$, $m_{DH} = m_\Delta = \Sigma(-1)^p\, m^p$



Discrete version (toric)

$$m^\delta_\Delta = \Sigma\,(-1)^p\, m^\delta_{\Delta_p} \underbrace{+\ \nu_p^w - \nu_p}_{\text{to avoid bdy cancellation.}}$$

Fact: In general, $T \curvearrowright M \longrightarrow \mathfrak{t}^*$

$$m_{DH} = \sum_{p\in M^T} (-1)^p\, m^p .$$